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On an integral with modified Bessel function

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Abstract. Using two different methods of calculation of the imaginary part of some Feynman integral, we obtain an explicit formula for a definite integral (over z from 0 to a) with modified Bessel function $I_0(z)$ and weight function $w(z)=z\cosh(AZ)/Z$, where $Z=\sqrt{a^2-z^2}$, and A and a are real parameters.

Let us consider the following Feynman integral:

$$V(s,q^2,p^2)\equiv -\mathrm{i}\int \frac{f((k+p)^2)g((k+p)^2)\,\mathrm{d}^4k}{(k+p+q)^2k^2} \tag{1a}$$

where $s\equiv(p+q)^2$; consider also the following functions f and g:

$$f(k^2)=\int_0^\infty \phi(\alpha)\exp(\alpha k^2)\,\mathrm{d}\alpha \qquad g(k^2)=\int_0^\infty \psi(\alpha)\exp(\alpha k^2)\,\mathrm{d}\alpha. \tag{1b}$$

This integral corresponds to a box diagram for forward scattering (see figure 1) in the theory of massless scalar fields in four-dimensional Minkowski space (one can think about ϕ^3-theory, but it is possible to have the same problem in the theory of a free scalar field which interacts with a photon field). We are interested in the discontinuity of V, which is defined as follows:

$$\mathrm{Disc}\,V(s,q^2,p^2)\equiv\lim_{\varepsilon\to 0}\{V(s+\mathrm{i}\varepsilon,q^2,p^2)-V(s-\mathrm{i}\varepsilon,q^2,p^2)\} \tag{2}$$

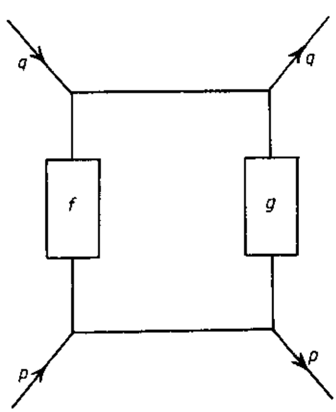

Figure 1. Box diagram of forward scattering for integral (1).

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for

$$q^2 = -Q^2 < 0 \qquad p^2 = -P^2 < 0 \qquad s > 0. \tag{3}$$

The main idea of the paper is that one could calculate such integrals in two different ways: with the help of the Cutkosky rule (see Cutkosky 1959) and using the double Borel transformation (see Nesterenko and Radyushkin 1982). Let us first describe the Cutkosky rule. As is known (see e.g. Berestezky *et al* 1980) this rule is valid just for diagrams we consider where the so-called cut crosses only two lines (see figure 2). In order to obtain Disc V one should change the propagators $S(p^2) = p^{-2}$ of these lines by their discontinuities, i.e. by $(-2\pi\mathrm{i})\delta(p^2)$:

$$\mathrm{Disc}\ V(s, q^2, p^2) = 4\pi^2\mathrm{i} \int \delta((k+p+q)^2)\delta(k^2)f((k+p)^2)g((k+p)^2)\,\mathrm{d}^4k. \tag{4}$$

Integration here could be performed rather simply if one were to use the centre-of-mass frame for two momenta p and q:

$$(p+q)_0 = \sqrt{s} \qquad \boldsymbol{p} + \boldsymbol{q} = 0 \tag{5a}$$

where

$$p_0 = \frac{\nu + p^2}{\sqrt{s}} \qquad \boldsymbol{p} = l\boldsymbol{n} \tag{5b}$$

$$q_0 = \frac{\nu + q^2}{\sqrt{s}} \qquad \boldsymbol{q} = -l\boldsymbol{n} \tag{5c}$$

$$\nu = (pq) = \frac{s - p^2 - q^2}{2} \qquad l = \sqrt{\frac{D}{s}} \qquad D = \nu^2 - p^2q^2 \tag{5d}$$

$$\boldsymbol{n} = (0, 0, 1). \tag{5e}$$

The result of the integration in (4) is ($\bar{\xi} \equiv 1 - \xi$)

$$\mathrm{Disc}\ V(s, q^2, p^2) = \frac{2\pi^3\mathrm{i}}{\sqrt{D}} \int_0^\infty \mathrm{d}\alpha \int_0^1 \mathrm{d}\xi\, \phi(\alpha\xi)\psi(\alpha\bar{\xi}) \exp(-\alpha\nu) \sinh(\alpha\sqrt{D}). \tag{6}$$

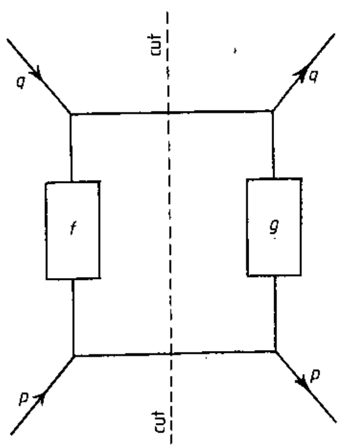

Figure 2. Cutkosky rules cut in the s-channel for the box diagram.

We wish to note at this point that for the whole amplitude of forward scattering we have a dispersion relation. But the amplitude V is only part of this whole amplitude and for this reason this dispersion relation takes the form:

$$V(s, q^2, p^2) = \int_{s_0}^{\infty} \frac{\rho_1(s')\,\mathrm{d}s'}{s'-s} + \int_{u_0}^{\infty} \frac{\rho_2(u)\,\mathrm{d}u}{u+s} + (\text{subtraction terms}) \tag{7}$$

where 'subtraction terms' means polynomials in s and are introduced in order to provide convergence of the integrals. Then in accordance with (2) the discontinuity of V could be expressed in terms of spectral densities ρ_i:

$$\mathrm{Disc}\ V(s, q^2, p^2) = 2\pi\mathrm{i}(\rho_1(s)\theta(s) - \rho_2(-s)\theta(-s)). \tag{8}$$

Let us now define the double Borel transformation. The single Borel transformation ($\tilde{B}$) I define in a slightly different form from the traditional (B):

$$\tilde{B}\left(s \to \frac{1}{\sigma}\right)[f(s)] = \frac{1}{\sigma} B\left(s \to \frac{1}{\sigma}\right)[f(s)] = \lim_{n\to\infty} \frac{(-1)^n}{\sigma^{n+1}(n-1)!} \frac{\mathrm{d}^n f}{\mathrm{d}s^n}\bigg|_{s=n/\sigma}. \tag{9}$$

Very important for applications of this transformation are the following properties:

$$\tilde{B}\left(s \to \frac{1}{\sigma}\right)\left[\frac{1}{s+a}\right] = \tilde{B}\left(s \to \frac{1}{-\sigma}\right)\left[\frac{1}{s-a}\right] = \exp(-\sigma a) \tag{10a}$$

$$\tilde{B}\left(s \to \frac{1}{\sigma}\right)\Big[\exp(-as)\Big] = \tilde{B}\left(s \to \frac{1}{-\sigma}\right)[-\exp(as)] = \delta(\sigma - a). \tag{10b}$$

These formulae are valid for $\sigma a > 0$. The double Borel transformation is then ($s > 0$)

$$\tilde{B}^2_{\mp}(f(s)) = \tilde{B}\left(\sigma \to \frac{1}{\mp s}\right)\left[\tilde{B}\left(z \to \frac{1}{\sigma}\right)[f(z)]\right] \tag{11}$$

and since $\tilde{B}[\text{polynomial}] = 0$ one has from (7) and (8)

$$\mathrm{Disc}\ V(s, q^2, p^2) = 2\pi\mathrm{i}\tilde{B}^2_{-}(V(s, q^2, p^2)). \tag{12}$$

Before direct application of the Borel transform it is useful to introduce in (1) an α-representation and to perform integration over momentum k:

$$V(s, q^2, p^2) \equiv \pi^2 \int_0^\infty \int_0^\infty \int_0^\infty \frac{\alpha\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\rho}{\mathscr{D}^2} \exp\left(\frac{\beta\rho s + \Delta}{\mathscr{D}}\right) \int_0^1 \phi(\alpha\xi)\psi(\alpha\bar{\xi})\,\mathrm{d}\xi \tag{13a}$$

$$\mathscr{D} = \alpha + \beta + \rho \tag{13b}$$

$$\Delta = \alpha(\rho q^2 + \beta p^2). \tag{13c}$$

The first Borel transformation $\tilde{B}(s \to 1/\sigma)$ acts only on $\exp(s\beta\rho/\mathscr{D})$ and gives therefore $(-\delta(\sigma + \beta\rho/\mathscr{D}))$. This delta-function allows one to reduce the number of integrations:

$$\tilde{B}_\sigma V(s, q^2, p^2) = -\pi^2 \int_0^\infty \alpha\,\mathrm{d}\alpha \int_0^1 \phi(\alpha\xi)\psi(\alpha\bar{\xi})\,\mathrm{d}\xi \int_0^1 \mathrm{d}y \frac{\exp(-\tau Q^2)}{\tau - \sigma} \exp\left(\frac{Q^2 y}{1-y}\frac{\tau\sigma}{\tau-\sigma}\right) \tag{14}$$

where $\tau \equiv \alpha y$. The second Borel transform acts only on ($R \equiv Q^2 y/\bar{y}$)†

$$\frac{1}{\tau - \sigma}\exp\left(\frac{Q^2 y}{1-y}\frac{\tau\sigma}{\tau-\sigma}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{-\infty}^{\infty} \mathrm{d}k_2 \exp((k_1^2 + k_2^2)(\sigma - \tau) + 2k_2\tau\sqrt{R} - R\tau) \tag{15}$$

† This trick has been invented by V A Nesterenko and A V Radyushkin.

and the result of this action is

$$\tilde{B}\left(\sigma\to\frac{1}{s}\right)\left[\frac{1}{\tau-\sigma}\exp\left(\frac{Q^2y}{1-y}\frac{\tau\sigma}{\tau-\sigma}\right)\right]=-I_0(2\alpha\sqrt{Q^2sy\bar{y}})\exp(-\alpha(sy+Q^2\bar{y})). \tag{16}$$

Here $I_0(z)$ is the modified Bessel function and is equal to

$$I_0(z)=\int_0^{2\pi}\frac{d\phi}{2\pi}\exp(z\cos\phi). \tag{17}$$

So one can obtain the result of application of the double Borel transformation to (1) and then, taking into account (12), can write

$$\begin{aligned}\text{Disc } V(s,q^2,p^2)\\ &=2\pi^3\mathrm{i}\int_0^\infty \alpha\,d\alpha\int_0^1 \phi(\alpha\xi)\psi(\alpha\bar{\xi})\,d\xi\\ &\times\int_0^1 dy\,I_0(2\alpha\sqrt{Q^2sy\bar{y}})\exp\{-\alpha[(s+P^2)y+Q^2\bar{y}]\}.\end{aligned} \tag{18}$$

Comparing (6) and (18) and taking into account the arbitrariness of functions $\phi(\alpha)$ and $\psi(\alpha)$ (the most simple way to confirm this is to choose $\phi(\alpha)=\psi(\alpha)=\delta(\alpha-\beta/2)$) gives

$$\int_0^1 dy\,I_0(2\beta\sqrt{Q^2sy\bar{y}})\exp\{-\beta[(s+P^2)y+Q^2\bar{y}]\}=\frac{\exp(-\beta\nu)\sinh(\beta\sqrt{D})}{\beta\sqrt{D}}. \tag{19}$$

Introducing the new integration variable $z=2a\sqrt{y\bar{y}}$ ($a\equiv\beta\sqrt{Q^2s}$) and bearing in mind that $D=(\nu-Q^2)^2+Q^2s$, one obtains

$$\int_0^a I_0(z)\cosh(A\sqrt{a^2-z^2})\frac{z\,dz}{\sqrt{a^2-z^2}}=\frac{\sinh(a\sqrt{1+A^2})}{\sqrt{1+A^2}}. \tag{20}$$

Here $A=(\nu-Q^2)/\sqrt{Q^2s}|_{P^2=0}=(1-2x)/(2\sqrt{x(1-x)})$, where $x=Q^2/2\nu$. For $P^2=0$ the scaling variable x varies from 0 to 1 and A from $-\infty$ to $+\infty$. Therefore formula (20) is valid for all real values of a and A. One can also rewrite (20) in another form, closer to those of Gradshtein and Ryshik (1980):

$$\int_0^\infty I_0\left(\frac{A}{\cosh(z)}\right)\cosh(B\tanh(z))\frac{dz}{\cosh^2(z)}=\frac{\sinh(\sqrt{A^2+B^2})}{\sqrt{A^2+B^2}}. \tag{21}$$

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